DOCUMENT RESUME

ED 175 693 SE 028 681

AUTHOR	Schaaf, William L., Ed.
TITLE	Reprint Series: Nature and History of Pi. RS-6.
INSTITUTION	Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	67
NOTE	52p.: For related documents, see SE 028 676-690
EDRS PRICE	MF01/PC03 Plus Postage.
DESCRIPTORS	Curriculum: *Enrichment: *History: *Instruction: Mathematics Education: *Number Concepts: *Probability: Secondary Education: *Secondary School Mathematics: Supplementary Reading Materials
IDENTIFIERS	*School Mathematics Study Group

ABSTRACT

This is one in a series of SMSG supplementary and enrichment pamphlets for high school students. This series makes available expository articles which appeared in a variety of mathematical periodicals. Topics covered include: (1) the history of the number pi: (2) what's new about pi: (3) the number pi: (4) pi and probability: and (5) from the Great Pyramid to Eniac. (MP)

Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which do not find a place in the curriculum simply because of lack of time, even though they are well within the grasp of secondary school students.

Some classes and many individual students, however, may find time to pursue mathematical topics of special interest to them. The School Mathematics Study Group is preparing pamphlets designed to make material for such study readily accessible. Some of the pamphlets deal with material found in the regular curriculum but in a more extended manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum.

This particular series of pamphlets, the Reprint Series, makes available expository articles which appeared in a variety of mathematical periodicals. Even if the periodicals were available to all schools, there is convenience in having articles on one topic collected and reprinted as is done here.

This series was prepared for the Panel on Supplementary Publications by Professor William L. Schaaf. His judgment, background, bibliographic skills, and editorial efficiency were major factors in the design and successful completion of the pamphlets.

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PREFACE

Probably no symbol in mathematics has evoked as much mystery, romanticism, misconception and human interest as the number pi (π). For most of us, our first meeting with this strange number, at once naive yet forbidding, is in our study of geometry. Here we learn that π is the ratio of a circumference to its diameter (C/D). But we soon find that π has many other significant properties, none of which has anything to do, directly, with the geometry of the circle. As we become more familiar with higher mathematics, we find, perhaps to our surprise, that the number π appears in the most unexpected places in algebra, in analysis, in the theory of numbers, in probability theory, and in various other branches of mathematics. Indeed, the noted English mathematician and master of paradox, Augustus de Morgan, once referred to "this mysterious 3.14159 ··· which comes in at every door and window, and down every chimney."

It is true that the numerical value of π as a measure of the circumference to the diameter is of some practical value. But the significance of the number π for theoretical mathematics goes far beyond its utilitarian value. It is no exaggeration to say that the familiar definition of π as C/D is in reality a bit of an accident, for the meaning and concept of π enters into mathematics in many ways. This is illustrated by an anecdote related by W. W. R. Ball in the following passage quoted from his *Mathematical Recreations and Essays:**

> De Morgan was explaining to an actuary what was the chance that a certain proportion of some group of people would at the end of a given time be alive, and quoted the actuarial formula, involving π, which, in answer to a question, he explained stood for the ratio of the circumference of a circle to its diameter. His acquaintance, who had thus far listened to the explanation with interest, interrupted him and exclaimed, "My dear friend, that must be a delusion: what can a circle have to do with the number of people alive at the end of a given time?"

During the three thousand or more years that mathematicians concerned themselves about the value and nature of π, much energy and amazing patience were manifested. Many of these activities led only to blind alleys; methods of investigation were abandoned as newer and

* Ball, W. W. R. and Coxeter, H. S. M. Mathematical Recreations and Essays, Macmillan, Eleventh Edition, p. 338.

more powerful mathematical tools became available. But these efforts were by no means altogether in vain. Not only did they stimulate theoretical mathematical discoveries, but also shed much light upon the computational aspects of mathematics. Indeed, the story of the determination of the numerical value of π is sufficiently exciting to form the subject of a separate pamphlet in this series. The present collection of essays emphasizes geometric and analytic aspects of the number π, although some allusions to its numerical value are unavoidable.

—William L. Schaaf

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ACKNOWLEDGMENTS

The SCHOOL MATHEMATICS STUDY GROUP takes this opportunity to express its gratitude to the authors of these articles for their generosity in allowing their material to be reproduced in this manner: David A. Lawson, who at the time his article was first published, was associated with the University of New Mexico; Phillip S. Jones, who, when his paper was first published, was associated with the University of Michigan; H. von Baravalle, who was then associated with Adelphi College in Garden City, New York; Walter H. Carnahan, of Purdue University; and Panos D. Bardis, of the University of Toledo.

The SCHOOL MATHEMATICS STUDY GROUP is also pleased to express its sincere appreciation to the several editors and publishers who have been kind enough to allow these articles to be reprinted, namely:

THE MATHEMATICS TEACHER:

(1) Walter H. Carnahan, *"Pi and Probability,"* vol. 46 (Feb. 1953), p. 65-66.

(2) Phillip S. Jones, *"What's New About π?,"* vol. 43 (March 1950), p. 120-122.

(3) H. von Baravalle, *"The Number π."* vol. 45 (May 1952), p. 340-348.

THE PENTAGON:

(1) David A. Lawson, *"The History of the Number Pi (π),"* vol. 4, (Fall 1944-Spring 1945), p. 15-24.

SCHOOL SCIENCE AND MATHEMATICS:

(1) Panos D. Bardis, *"Evolution of π: An Essay in Mathematical Progress from the Great Pyramid to Eniac,"* vol. 60 (Jan. 1960), p. 73-78.

FOREWORD

The literature dealing with the number π is indeed voluminous. Many aspects of this familiar constant have been discussed and many properties described—its history; its uses; its numerical approximation; its characteristics as a number, whether rational or irrational, algebraic or transcendental; the distribution of its digits; its relation to the number e, its relations to probability theory, to the theory of numbers, and to other mathematical topics.

The number π has been the subject of doggerel verse designed as a mnemonic to remember the sequence of digits in its approximation; and both π and e have been the themes of more serious poetry.

The word "pi" has been used in naming mathematics clubs. With tongue in cheek, one writer has titled an article: "How to Make Pi Digestible."

Be that as it may, this first essay presents the reader with a general, overall picture concerning π, thereby setting the stage for the other essays. In so doing, it was impossible to avoid anticipating a number of items which will be discussed in greater detail in a subsequent article. It is to be hoped that such occasional repetition will not prove irksome.

The History of the Number Pi

David A. Lawson

The easiest and probably the earliest area computed by man was the area of the square. So it was only natural that attempts to find the area of a circle, which presented a far more difficult problem, gave rise to the idea of considering a square of equal area. This idea developed into one of the classical problems of geometry, "the squaring of the circle." It is found that much of the early history of the number represented by π is connected with this problem.

The Egyptians had a method for finding the area of a circle by comparing it with a square. The rule as presented in the *Rhind Papyrus* assumed that the area of a circle is equivalent to that of a square whose side is eight-ninths of the diameter of the circle. This amounts to stating that the area of a circle of radius R is $(256/81)R^2$. Upon comparison with the true area, πR^2, it may be seen that the ancient Egyptian rule is equivalent to having $\pi = 256/81 = 3.16049+$. This is not a very close approximation to the true value of π, yet it is closer than other approximations obtained before the time of the Greek mathematicians.

While the value of π was obtained by the Egyptians as a result of their attempts to "square the circle," the Babylonians, on the other hand, were interested in the rectification of the circumference; that is, in finding directly the relationship between the radius and the circumference. The Babylonians reached the conclusion that the circumference of a circle is equal to a line which is "a little more than six times the radius."

The Hebrews considered the circumference of a circle as equal to three times the diameter. This may be seen in at least two places in the Old Testament, 2 Chronicles 4:2, and 1 Kings 7:23. The first of these two verses is as follows: "Also he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits did compass it about."

The problem of squaring the circle was a problem which the Greeks took up with zest the moment they realized its difficulty. Although many Greek mathematicians and nonmathematicians became interested, the contributions of several men stand out. First, Antiphon of Athens must be mentioned. "Antiphon inscribed within a circle some one of the

regular inscribed polygons which can be inscribed. On each side of the inscribed polygon as a base he described an isosceles triangle with its vertex on the arc of the segment of the circle subtended by the side. This gave him a regular inscribed polygon with double the number of sides. Repeating the construction with the new polygon, he had an inscribed polygon with four times as many sides as the original polygon. Continuing the process, Antiphon thought that in this way the area of the circle would be used up, and he would some time have a polygon inscribed in the circle the sides of which would, owing to their smallness, coincide with the circumference of the circle" [4, p. 222]*. Antiphon assumed that he could make a square equal in area to any regular polygon, an impossible assumption. Otherwise, his method is still an approximation.

Antiphon started the "ball rolling" in the right direction and soon afterward Bryson of Heraclea gave it another push. His method was similar to that of Antiphon but with the addition of a circumscribed polygon. Bryson was the first to introduce into mathematics "the concept of upper and lower limits in approximations" [8, p. 125], comparing a circle with its regular inscribed and circumscribed polygons. By using a modification of Bryson's method, Archimedes was later able to calculate his approximation to π.

It is interesting to note that Euclid, in his Elements, made no effort to find the area of a circle or to calculate the ratio of the circumference to the radius.

Following Euclid there lived "the greatest mathematician of antiquity" [6, p. 166], Archimedes. In using the method originated by Bryson, Archimedes made one important change; he considered the perimeters of the polygons and the radius of the circle rather than the area. This method for finding the limits between which π must lie was practically the only one used for about two thousand years preceding the invention of the differential calculus. Archimedes found the value of π to lie in the range.

$$22/7>\pi>223/71.$$

Later he made even a better approximation, his figures giving

$$195882/62351>\pi>211872/67441,$$

or

$$3.1416016>\pi>3.1415904.$$

The arithmetic mean between these two limits gives the close approximation 3.141596.

* Numbers in brackets refer to the literature cited at the end of this paper.

Archimedes' calculations were most remarkable considering the "unwearied perseverance" [8, p. 127] he must have employed to get such results by using the crude system of Greek notation. Anyone who is familiar with the Greek system of numbers will agree that even the calculation of π to 707 decimal places is much less a wonder than Archimedes' results, correct to only four decimal places.

One other result obtained by the Grecian school might be mentioned. The astronomer, Ptolemy, who lived in Alexandria about 150 A.D., expressed π as the sexagesimal fraction $3 + 8/60 + 30/3600$, or $3.14166\cdots$.

The Romans added nothing to the work of the Greeks. Instead, they seemed to have lost much of the exactness which Archimedes had contributed. Even though the Romans seemed to realize that 3 1/7 was closer to the true value of π than $3\frac{1}{8}$, they frequently employed the latter fraction because it was "more convenient" [1, p. 351].

A Roman treatise on surveying contains the following instructions for squaring the circle: "Divide the circumference of a circle into four parts and make one part the side of a square; this square will be equal in area to the circle" [8, p. 128]. Although this is actually an impossible construction, if the construction were possible, π is found to be equal to 4. This is more inexact than any other known computation for the number π.

For the thousand years following the decline of the Greeks, the center of mathematical activity shifted eastward. The Hindus, in particular, were very active during this period. Their mathematicians carried the method of Archimedes far enough to get an answer closer to the true value than either Archimedes or Ptolemy. In spite of the fact that Aryabhatta (about 500 A.D.) calculated π correctly to at least four decimal places, the great Hindu mathematician Brahmagupta gave the value $\sqrt{10}$ which equals 3.16228. Unfortunately, it was this latter value for π which spread to Europe and was used quite extensively during the middle ages.

The Chinese mathematician Tsu Ch'ungchih (fifth century A.D.) should not be overlooked. Probably by using the method of Archimedes, he found that the true value of π lies between 3.1415926 and 3.1415927.

The Arabians must be remembered in view of the fact that they handed down the results of the Greek and Hindu mathematicians to the awakening countries of Europe. In this way many of these results were probably preserved. The Arab scholar, Muhammed Ibn Musa Alchwarizmi, who brought the principles of our present system of numerical notation from India and introduced it to the Mohammedan world, brought together the various Greek and Hindu approximations for the number π.

Going back to Europe, it is found that little was done in mathematics

during the Dark Ages. The value of π was calculated on more than one occasion, but all these results were less accurate than those of the Greeks and Hindus. For example, Michael Psellus was a scholar who lived in the latter part of the eleventh century. Although his contemporaries called him "first of philosophers," what survives of his mathematical work is very inaccurate. "In a book purporting to be by Psellus on the four mathematical sciences, arithmetic, music, geometry, and astronomy, the authors' favorite method to find the area of a circle is given. The area was taken as the geometric mean between the inscribed and circumscribed squares; this gives a value to π equal to the square root of 8, or 2.8284271" [5, p. 545]. "The greatest mathematical genius of the middle ages" [9, p. 395], Leonardo of Pisa (thirteenth century), was able to get a little closer in his calculations; he gave π equal to the value 3.1418.

During the fifteenth century the sciences began to revive. Greater interest was shown in mathematics, and especially, at first, in the quadrature of the circle. This interest was, to a large extent, aroused by Cardinal Nicolas de Cusa who claimed to have discovered a method for squaring the circle. None doubted that the cardinal had solved this famous problem until his construction was proved false by Regiomontanus.

For the next couple of hundred years the circle-squarers as well as the calculators were very active. But, during this period, the reputable mathematicians began to realize that the ancient problem of quadrature was an impossibility. They wasted little time upon it except to show that the results of the various circle-squarers were incorrect. Of course, these demonstrations had little effect on the circle-squarers. "In the future as in the past, there will be people who know nothing of this demonstration and will not care to know anything, and who believe that they cannot help succeeding in a matter in which others have failed, and that just they have been appointed by Providence to solve the famous puzzle" [8, p. 116].

A few years after the revival of interest in mathematics, or about 1500 A.D., mathematicians began to extend the value of π to more places of decimals. Adrian Metius published his value of π correct to six decimal places, and Vieta, in 1579, calculated the value correct to nine decimal places. In 1593, Adrian Romanus determined π to 15 places, but in order to do so he had to calculate the perimeter of an inscribed regular polygon of 2^{30} sides, where

$$2^{30} = 1{,}073{,}741{,}824.$$

All these results were outdone by Ludolf Van Ceulen who carried Archimedes' method to a calculation of π correct to 35 decimal places. He was so proud of his work that he requested in his will that his results

be engraved upon his tombstone. Hermann Schubert tells us that in honor of Ludolf π is called today in Germany the Ludolfian number. The history of Archimedes' method of calculating π was closed in 1630 when Grienberger, the last to employ the method, announced his result correct to 39 places of decimals.

A new period in the solution of our problem began in the second half of the seventeenth century with the development of the calculus. New analysis came to the aid of the investigators, and the method of Archimedes became obsolete. The new methods attempted to express π analytically by developing it as an infinite product or series. The first important new result was produced by John Wallis (1616–1703) who proved the two relationships,

$$\pi/2 = 2/1 \cdot 2/3 \cdot 4/3 \cdot 4/5 \cdot 6/5 \cdot 6/7 \cdot 8/7 \cdot 8/9 \cdots,$$

and

$$\frac{4}{\pi} = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \cfrac{81}{2 + \cdots}}}}}.$$

The continued fraction form had previously been expressed without proof by Lord Brouncker (1620–84).

The first infinite series developed for the study of the circle was the series,

$$\pi/4 = 1 - 1/3 + 1/5 - 1/7 + 1/9 - 1/11 + \cdots.$$

Although others knew it previously, this series was published by Leibnitz and bears his name. The Leibnitz series converges so slowly as to be inconvenient in practice. It is the series obtained from the expansion or arctangent x.

$$\arctan x = x - x^3/3 + x^5/5 - x^7/7 + \cdots,$$

when x is set equal to 1.

If x is taken equal to $\sqrt{1/3}$, the arctangent series becomes

$$\pi/6 = \sqrt{1/3} \cdot (1 - 1/3 \cdot 3 + 1/3^2 \cdot 5 - 1/3^3 \cdot 7 + 1/3^4 \cdot 9 - 1/3^5 \cdot 11 + \cdots),$$

a series which converges much more rapidly. This general series was discovered by James Gregory independently of Liebnitz. The series is frequently called Gregory's series.

By using various infinite series, the following men extended the value

of π to more and more decimal places during the next two hundred years [1, pp. 356-7]:

Abraham Sharp, in 1699, to 71 correct decimal places;
Machin, about 1706, to 100 correct decimal places;
De Lagney, in 1719, to 122 correct decimal places;
Vega, in 1789, to 126 correct decimal places;
Vega, in 1794, to 136 correct decimal places;
Rutherford, in 1841, to 152 correct decimal places;
Dase, in 1844, to 200 correct decimal places;
Clausen, in 1847, to 248 correct decimal places;
Rutherford, in 1853, to 440 correct decimal places;
William Shanks, in 1873, to 707 decimal places.

What about the circle-squarers while all this was going on? Of course they were as busy as ever. But, at the same time, various mathematicians were trying to prove that the quadrature of the circle is an impossibility. The first step was made in 1761 by the French mathematician Lambert who proved that π is not a rational number. In 1794, Legendre showed that π cannot be the root of a quadratic equation with rational coefficients. "This definitely disposed of the question of squaring the circle, without, of course, dampening in the least the ardor of the circle-squarers" [3, p. 117].

The intimate connection between the number e and π had been well known for some years; so when, in 1873, Hermite proved that e was transcendental, the efforts were redoubled to prove π was also a transcendental number. Nine years later, Professor Lindeman of Freiburg, Germany, was successful in proving this fact.

We are so accustomed to the use of the symbol π to express the ratio of the circumference of a circle to the diameter that we are in danger of overlooking the fact that the use of the symbol π is quite recent. It was apparently used in this connection by William Jones in 1706. But it was Euler, "the most prolific mathematical writer who ever lived" [6, p. 168], who made this symbol popular by using it consistently after 1737.

The number π has properties of which many of us are unaware. This is especially true in the field of probability. An interesting experiment was conducted by Professor Wolff of Zurich some years ago. "The floor of a room was divided up into equal squares, so as to resemble a huge chessboard, and a needle exactly equal to the side of these squares was cast haphazardly upon the floor. If we calculate, now, the probabilities of the needle so falling as to lie wholly within one of the squares, that is, so that it does not cross any of the parallel lines forming the squares, the result of the calculation for this probability will be found to be exactly equal to $\pi - 3$. Consequently, a sufficient number of casts of the

needle according to the law of large numbers must give the value of π approximately. As a matter of fact, Professor Wolff, after 10,000 trials, obtained the value of π correct to 3 decimal places" [8, p. 140].

There have been other methods of this type employed to calculate π. For example, if two numbers are written down at random, it has been found that the probability that they will be prime to each other is $6/\pi^2$. "Thus, in one case where each of 50 students wrote down 5 pairs of numbers at random, 154 of the pairs were found to consist of numbers prime to each other. This gives $6/\pi^2 = 154/250$, from which we get $\pi = 3.12$" [1, p. 359].

Let us consider the question of benefits which might be derived from calculating the value of π to a large number of decimal places. Such calculations show the power of modern methods compared with some of the older ones. But, for practical use, the general opinion seems to be that there is no need to have the value of π to more than 10 or 15 decimal places. Measurements are seldom correct to as many as 10 decimal places, and if π is used to many more places, the result would have fictitious accuracy.

In 1899, Hermann Schubert gave an example "to show that the calculation of π to 100 or 500 decimal places is wholly useless. Imagine a circle to be described with Berlin as centre, and the circumference to pass through Hamburg; then let the circumference of the circle be computed by multiplying its diameter by the value of π to 15 decimal places, and then conceive it to be actually measured. The deviation from the true length in so large a circle as this even could not be as great as the 18 millionth part of a millimetre" [8, p. 398]. Some years ago the late Professor Newcomb remarked, "Ten decimals are sufficient to give the circumference of the earth to the fraction of an inch, and thirty decimals would give the circumference of the whole visible universe to a quantity imperceptible with the most powerful microscope" [9, p. 398].

Curiously, attempts have been made to "fix" the value of π by law. Typical of these attempts was the bill presented to the legislature of Indiana in 1897. The bill was suggested by a local circle-squarer who said that "the present rule in computing the circle's area is entirely wrong." The bill was introduced as "A bill for an act introducing a new mathematical truth and offered as a contribution to education to be used only by the state of Indiana free of cost by paying any royalties whatever on the same, provided it is accepted and adopted by the official action of the legislature of 1897." The bill was considered by the Committee on Education which recommended that it "do pass." The bill passed the house but was lost in the state senate.

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FOREWORD

In this essay the author presents a number of interesting sidelights on the number π from earliest times as well as from today.

Some of the notation used may be unfamiliar. For the reader who may have forgotten, we recall that the symbol n! (read "factorial n") means the continued product

$$n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$$

We also note that the limiting value of the expression

$$\left(1+\frac{1}{k}\right)^k$$

as k becomes indefinitely great, is designated by the letter e. Using the binomial theorem, it can be shown that

$$\lim_{k\to\infty}\left[\left(1+\frac{1}{k}\right)^k\right]^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots,$$

or

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots.$$

When $x = 1$, we have

$$e = 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots.$$

Calculation shows that e is approximately equal to $2.7182818\cdots$ It is known that e is both irrational and transcendental.

The number e plays a very significant role in mathematical analysis. Thus it is used as the base for the system of *natural logarithms*. (Ordinary logarithms to the base 10 are called common or Briggian logarithms.) The conversion from one system to the other can be effected by using the relation

$$\frac{\log_e N}{\log_e 10} = \log_{10} N,$$

where $\log_e 10 = 2.3025\cdots$ and $\frac{1}{\log_e 10} = .43429\cdots$.

For the sake of clarity, the base is usually indicated. For example, $\log_{10} 100 = 2$; $\log_e e = 1$; etc. If the base to be used is not explicitly indicated, as in "log x," some ambiguity might arise. Hence if $\log_e x$ is meant, it is preferably written ln x, where "ln" indicates "natural logarithm."

What's New About Pi?

*Phillip S. Jones**

In January of 1948 a new footnote, if not a new chapter, was added to the history of π.[1] At this time John W. Wrench, Jr., of Washington, D. C., and D. F. Ferguson of Manchester, England, published jointly the corrected and checked value of π computed to 808 decimal places.[2] This concludes a project begun by Dr. Ferguson in 1945 when he became interested in the correctness of the unchecked 707 decimal place value first given by the Englishman William Shanks in 1873 and revised by Shanks himself in 1874.[3]

Ferguson found errors in Shanks' value beginning with the 528th place and gave a corrected value to 620 places. He had extended this to 710 places by January of 1947.

In this latter month Dr. Wrench collaborating with Levi B. Smith published an 808 decimal place value.[4] Shortly thereafter Ferguson discovered an error beginning with the 723rd place of Wrench's value. The final 808 place value published jointly by these two computers may be regarded with considerable confidence since they did their computations independently and using different formulas. Ferguson used the formula $\pi/4 = 3\tan^{-1} \frac{1}{4} + \tan^{-1} 1/20 + \tan^{-1} 1/1985$ which he obtained from R. W. Morris but which has been shown to have appeared in 1893 in S. L. Loney's *Plane Trigonometry*. Wrench used Machin's formula $\pi/4 = 4 \tan^{-1} 1/5 - \tan^{-1} 1/239$. This latter was also used by Shanks.

These are prodigious feats of computation and immediately raise the question of why should anyone undertake them. The famous American astronomer and mathematician Simon Newcomb once remarked, "Ten decimal places are sufficient to give the circumference of the earth to the

*Such is the incredible pace of technological progress that, less than ten years after this paper was written, the value of π was determined to more than 100,000 decimal places. This computation was carried out on July 29, 1961, on an I.B.M. electronic system, in less than nine hours. — *Editor*

[1] Prof. E. H. C. Hildebrandt originally suggested that *Miscellanea* include a note on the new value of π.

[2] "A New Approximation to π (Concluded)," *Mathematical Tables and Other Aids to Computation*, III, pp. 18–19.

[3] D. F. Ferguson, "Evaluation of π, Are Shank's Figures Correct?" Mathematical Note, 1889, *Mathematical Gazette* 30 (May, 1946), pp. 89–90.

[4] "A New Approximation to π," *Mathematical Tables and Other Aids to Computation*, II, p. 245.

fraction of an inch, and thirty decimals would give the circumference of the whole visible universe to a quantity imperceptible with the most powerful telescope," according to Kasner and Newman.[5] The latter then give two reasons for such calculations: the hope to find a clue to the transcendental nature of π, and "the fact that π, a purely geometric ratio, could be evolved out of so many arithmetic relationships — was a never ending source of wonder." The former could not have motivated our modern workers since π was proved irrational by J. H. Lambert in 1761 and transcendental by F. Lindeman in 1882. Shanks, however, might have had some such motivation and hence it may be of interest to quote his own words from his first publication on this subject. "Toward the close of the year 1850 the Author first formed the design of rectifying the circle to upwards of 300 places of decimals. He was fully aware at that time that the accomplishment of his purpose would add little or nothing to his fame as a Mathematician, though it might as a Computer; nor would it be productive of anything in the shape of pecuniary recompense at all adequate to the labour of such lengthy computations. — He was anxious to fill up his scanty intervals of leisure with the achievement of something original, and which, at the same time, should not subject him to either too great tension of thought or to consult books. — The Writer entertains the hope, that Mathematicians will look with indulgence on his present 'Contributions' to their favorite science, and also induce their Friends and Patrons of Mathematical Studies, to accord him their generous support by purchasing copies of the work." (The book was "Printed for the author" — i.e., privately published.) Later Shanks says, "— no one, so far as we know, has hitherto been able to — and we are of the opinion that it can never be accomplished — to ascertain the *limit*, strictly speaking of the ratio under consideration."[6]

Our modern computers have not published analyses of their motives. They appear to have been actuated by intellectual curiosity and the challenge of an unchecked and long untouched computation. However, it might be noted that lengthy and rapid computations and machines to perform them are of great interest these days. For example H. S. Uhler has computed $\frac{1}{2}\log \pi$, $\log \pi$, and $\ln \pi$ to 214 and 213 decimal places for the purpose of using them later in computing tables of $\ln x!$[7] Wrench has computed tables of $\pi^{\pm n}/n$ to 206 significant figures to be used in later calculations of $\pi^n/n!$ which in turn are needed in calculating certain transcendental functions.[8] Werner F. Vogel has computed *Angular Spac-*

[5] Edward Kasner and James Newman, *Mathematics and the Imagination* (New York: Simon and Schuster, 1940), p. 78.

[6] William Shanks, *Contributions to Mathematics Comprising Chiefly the Rectification of the Circle to 607 Places of Decimals* (London: 1853), pp. v, vi, xiv.

[7] *Mathematical Tables and Other Aids to Computation*, I, p. 55.

[8] *Ibid.*, I, p. 452.

ing Tables[9] for use in gearing problems which include tables giving angles in radians to ten decimal places. To compute these a many decimal place value of π was used. (He cites a 70 place value in the book.)

NOTES ON OLDER FACTS

Historical discussions of π and collections of interesting formulas for its calculation are to be found in many places,[10] but two interesting items in its long history are often inadequately treated.

It is frequently stated that the Egyptians calculated the area of a circle as $(8/9d)^2$ which is equivalent to giving a value to π of 3.1605. Though not incorrect, such a statement by stating truths in modern notation and too concisely fails to display several interesting features of the Egyptian procedure. Actually the Egyptian in each case subtracted from the diameter of the circle one-ninth of the diameter and then squared this result. This is consistent with the Egyptian use of unit fractions; the use of 8/9 is not. This second more exact statement also avoids any implication that the Egyptian had conceived of an abstract number π, a mathematical constant, in any modern sense. Further, the exact statement furnishes a plausible suggestion as to how the Egyptian arrived at his procedure. In the Rhind Papyrus the calculation of volumes of cylinders precedes the calculation of areas of circles. This fact has led A. B. Chace and others to speculate that the Egyptians may have made a circular cylindrical container and then several sizes of square prisms of the same height. They speculate that it was by comparing the liquid capacity of the cylinders and these prisms that the Egyptians determined experimentally that the prism erected on the square whose side was one-ninth less than the diameter of the cylinder most nearly approximated the volume of the cylinder.[11]

Another often quoted but rarely documented tale of π is that of the attempt to determine its value by legislation. House Bill No. 246, Indiana State Legislature, 1897, was written by Edwin J. Goodwin, M.D. of Solitude, Posey County. It begins as follows: "A bill for an act intro-

[9] Werner F. Vogel, *Angular Spacing Tables* (Detroit: Vinco Corp., 1943, $10.00).

[10] Kasner and Newman, *op. cit.*, pp. 65–79. D. E. Smith, "The History and Transcendance of π," in *Monographs on Topics in Modern Mathematics* (J. W. A. Young, Ed.). (Longmans, Green, 1915). pp. 389–416.

[11] Arnold Buffum Chace, *The Rhind Mathematical Papyrus* (The Mathematical Association of America, 1927), Vol. I, pp. 35–36, 86–88, 91–92. A summary, with references, of other theories which have been advanced to explain this Egyptian procedure may be found in J. L. Coolidge, *A History of Geometrical Methods*, (Oxford: 1940), p. 11.

ducing a new mathematical truth and offered as a contribution to education to be used only by the State of Indiana free of cost by paying any royalties whatever on the same —.

Section 1. Be it enacted by the General Assembly of the State of Indiana: It has been found that a circular area is to the square on a line equal to the quadrant of the circumference, as the area of an equilateral rectangle is to the square on one side. . ."

The bill was referred first to the House Committee on Canals and then to the Committee on Education which recommended its passage. It was pased and sent to the Senate where it was referred to the Committee on Temperance which recommended its passage. In the meantime the bill had become known and ridiculed in various newspapers. This resulted in the Senate's finally postponing indefinitely its further consideration in spite of the backing of the State Superintendent of Public Instruction who was anxious to assure his state textbooks of the use, free, of this copyrighted discovery. The detailed account of the bill together with contemporary newspaper comments makes interesting reading.[12]

[12] Donald F. Mela directed the writer's attention to the source for this data; namely, Will E. Edington, "House Bill No. 246, Indiana State Legislature, 1897" *proceedings of the Indiana Academy of Science*, Vol. 45 (1935), pp. 206–210. Thomas F. Holgate, "Rules for Making Pi Digestible" in the Contributor's Club of *The Atlantic Monthly* for July, 1935 is also pertinent.

FOREWORD

In this unusual essay, the author stresses certain geometric aspects of π, bringing together a number of interesting related ideas, particularly the three significant numbers π, e, and G, where e is the familiar limit of $(1 + 1/x)^x$ as x becomes infinitely great, and G is the ratio of the Golden Section, namely, $\frac{1}{2}(\sqrt{5} - 1)$. The reader who is familiar with the "Golden Measure" will recall that

$$G = \tfrac{1}{2}(\sqrt{5} - 1) = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{5}$$

is the positive root of the quadratic equation $x^2 + x - 1 = 0$, which is obtained when dividing a unit segment into mean and extreme ratio.

Not the least interesting features of Baravalle's essay are the geometric construction of the value of π, and Kochansky's geometric, approximate quadrature of the circle. For further observations on the quadrature of the circle, the reader is referred to the foreword to the essay by P. D. Bardis.

Indeed, approximate geometric constructions of π have been most ingenious. One of the most remarkable such constructions yields a straight line segment which differs from π by less than .0000003. The construction is quite simple and is described by Martin Gardner in the *Scientific American* for July, 1960, page 156.

The Number Pi

H. von Baravalle

Two outstanding constants of mathematics have been dealt with in previous articles in THE MATHEMATICS TEACHER, the number e, the base of the natural logarithms[1] and the number G, the ratio of the Golden Section.[2] To complete this series, the present article takes up the third and best known constant, the number π.

As its symbol indicates (π stands for periphery), it represents the ratio of the two outstanding dimensions of the circle, the way around it and the distance across it.

$$\pi = \frac{\text{circumference of a circle}}{\text{diameter of the circle}}.$$

Expressing the diameter in terms of the radius r, we obtain the formula for the circumference of the circle c:

$$\frac{c}{2r} = \pi;\ c = 2\pi r.$$

This is by far not the only ratio in which this constant appears. For instance, π is also the ratio of the area of a circle A to the area of the square erected on its radius r:

$$\pi = \frac{A}{r^2};\ A = \pi r^2.$$

It further appears in many other formulae. The volume V of a circular cylinder with a base-radius r and altitude h is:

$$V = r^2\pi h$$

and of a circular cone—

$$V = \frac{r^2\pi h}{3}$$

[1] December 1945 issue (Volume XXXVIII, No. 8).
[2] January 1948 issue (Volume XLI, No. 1).

The surface of a sphere is:

$$A = 4r^2\pi$$

and its volume—

$$V = \frac{4}{3}r^3\pi.$$

The domain of π also extends beyond circular structures. The area of an ellipse with the semi-axes a and b is

$$A = ab\pi$$

and the volume of an ellipsoid, with the three semi-axes, a, b, and c is

$$V = \frac{4}{3}abc\pi.$$

The area enclosed in a cardioid drawn in Figure 1 as an envelope of circles is

$$A = \frac{3}{2}a^2\pi,$$

in which a stands for the diameter of the circle whose circumference is indicated by the dotted line.[1]

Further examples of curves whose formulae contain π are the roses. The area enclosed by a three-leaved rose (black portions in Figure 2) is

$$A = \frac{1}{4}a^2\pi,$$

in which a stands for the radius of the circle circumscribed around it. The area enclosed by a four-leaved rose (black area in Figure 3) is

$$A = \frac{1}{2}a^2\pi,$$

in which a again denotes the radius of the circumscribed circle. The volume of a ring (torus), obtained by rotating a circle with radius a about an axis in the same plane at a distance of b units from the center of the circle is expressed in the following formula:

$$V = 2a^2b\pi^2.$$

The volume of the solid of rotation produced by rotating an astroid about one of its axes is:

[1] To construct Figure 1, the dotted circle is divided into thirty-two equal parts. Each of the thirty-two points of division becomes the center of a circle whose radius is its distance from the highest point on the circle (upper end of vertical diameter).

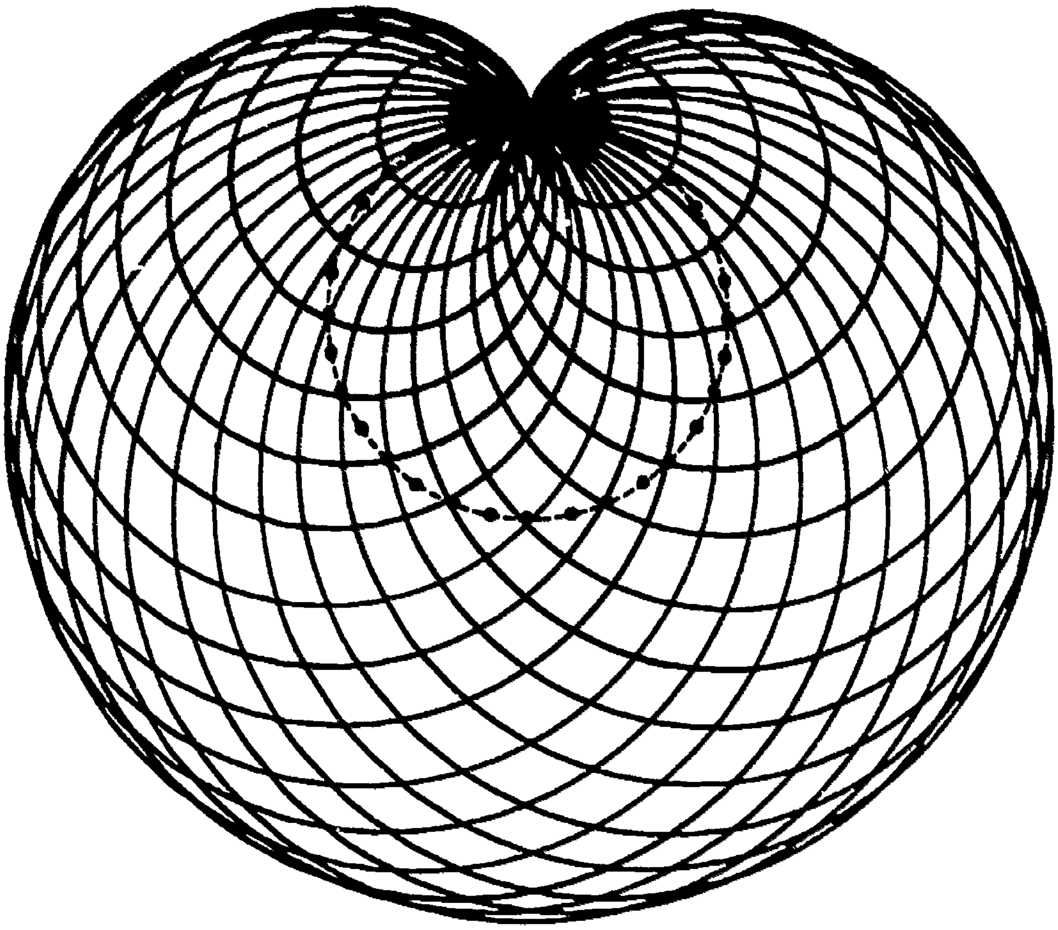
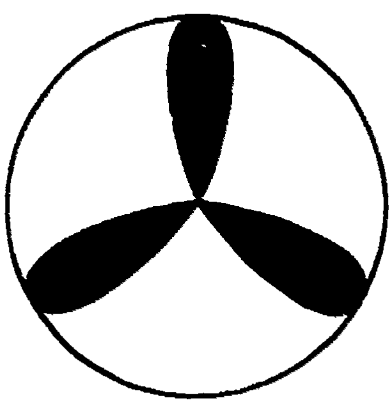


FIG. 1. The cardioid.



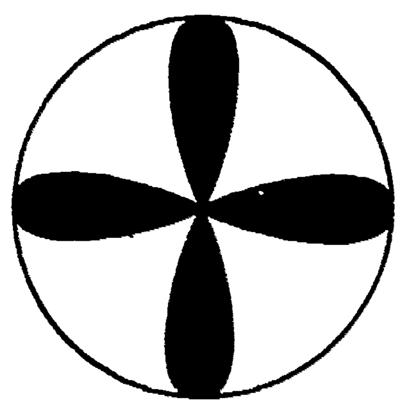


FIG. 2. The three-leaved rose. FIG. 3. The four-leaved rose.

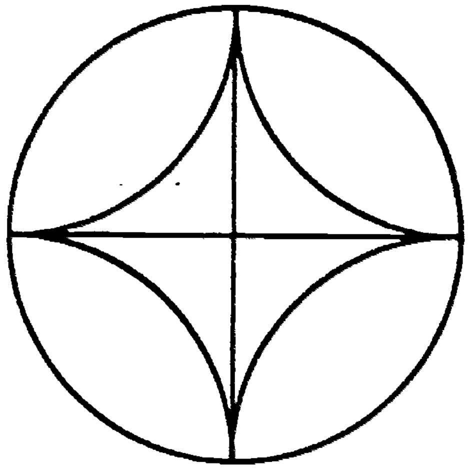


FIG. 4. The astroid.

$$V = \frac{32}{105} a^3\pi.$$

Here a represents the distance of any of the star points from the center, the radius of the circumscribed circle. The surface area of the same solid of rotation is:

$$A = \frac{12}{5} a^2\pi.$$

The formula for the volume of the solid $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$ whose traces on the coordinate planes are astroids, also contains π:

$$V = \frac{4}{35} a^3\pi.$$

All these formulae are obtained by integral calculus.

We can also go beyond areas, surfaces and volumes to find π again in a variety of relationships. A semicircle (radius r), cut out of sheet metal and balanced on a point, will be in equilibrium only when the point of support lies on its axis of symmetry at a distance d from the center, which is:

$$d = \frac{4r}{3\pi}.$$

π even appears in formulae of probability, statistics and in the field of an actuary.

Any vibration, mechanical, acoustical or electrical, proceeds with varying speed. By determining the distance covered by a point on a vibrating musical cord between its extreme positions during a certain time unit, we obtain its average speed of motion. The actual speed of the point is greater every time the cord is near to passing its middle position. It is less than the average speed every time the point finds itself

near to one of its extreme elongations. The maximum speed occurs when the point passes in either direction through its position of rest. This maximum speed is in any vibration exactly $\pi/2$ times the average speed.[4] As this holds good for vibrations accompanying every sound of our own vocal cords and in the air around us, π is contained in every word and sentence we say.

The value of π up to 22 decimal places is

3.1415926535897932384626 . . .

These successive numerals are the same as the number of letters contained in the successive words of the French verse:

"Que j'aime a faire apprendre
Un nombre utile aux sages.
Immortel Archimède, artiste ingénieur,
Oui de ton jugement pent priser la valeur."

Translation: "How I like to teach a number, useful to the learned: Immortal Archimedes, skillful investigator, yes, the number can tell the praise of your judgment."

Until recently π had been calculated to 707 decimal places. This figure had been obtained by an Englishman, William Shanks, in 1853. With the help of the modern electronic computing machines, the number of decimal places has now been extended to over 2,000.

The history of the number π dates back 3,500 years, as far as historical

[4] The differential equation of a vibration is

$$\frac{d^2x}{dt^2} = -a^2x$$

and its complete solution is $x = c \sin (at - \alpha)$. In this equation, c and a represent arbitrary constants. For $x = 0$ at $t = 0$, the solution is:

$$x = c \sin at.$$

sin at. Maximum speed:

$$\frac{dx}{dt} = ac \cos at; \quad \text{for } t = 0: \left.\frac{dx}{dt}\right|_{max} = a \cdot c.$$

Average speed:

$$\text{For} \quad at = \frac{\pi}{2}; \quad x = c \quad \text{and} \quad \frac{x}{t} = \frac{2ac}{\pi}.$$

The ratio.

$$\frac{\left.\frac{dx}{dt}\right|_{max}}{\frac{x}{t}}$$

is therefore: $\pi/2$.

records show. The Egyptian Rhind Papyrus, dating back as far as 1700 B. C., gives directions for obtaining the area of a circle. Expressed in modern symbols, its formula with A for the circle's area and d for its diameter, is as follows:

$$A = \left(d - \frac{1}{9}d\right)^2 = d^2\left(1 - \frac{1}{9}\right)^2$$
$$= 4r^2\left(\frac{8}{9}\right)^2 = r^2\frac{4 \cdot 64}{81} = r^2\frac{256}{81}.$$

The fraction 256/81, which here takes the place of π, equals in decimals 3.16050 ⋯. Compared with π, (3.14159 ⋯), the difference is 0.01891 ⋯, or less than 1/50.

Archimedes expresses π numerically as follows:

$$3\frac{1}{7} > \pi > 3\frac{10}{71}.$$

Expressed in decimals, the same relationships would read:

$$3.142857 \cdots > \pi > 3.140845. \cdots$$

Midway between these two values of Archimedes lies the number 3.141851, which, compared with π is only 0.000259 ⋯ or about 2½ ten-thousandths greater. In ancient China, π was expressed by Ch'ang Höng (125 A.D.) as $\sqrt{10} = 3.162 \cdots$, the accuracy of which is only slightly less than the value given in the Egyptian papyrus. In 265 A.D. Wang Fan expressed the value of π by the fraction 142/45, or 3.15555 ⋯. In 470 A.D. Ch'ung-chih gave a different fraction: 355/113, or 3.1415929 ⋯, which is correct all the way out to 6 decimal places. In India Aryabhata (510 A.D.) expressed π in this way: "Add 4 to 100, multiply by 8 and add 62,000. This is the approximate circumference of a circle whose diameter is 20,000." Thus π appears as the fraction 62832/20000, which resolves to 3.1416, and is less than one ten-thousandth off.

Though some of these values are sufficiently accurate to have met the practical demands of their times, none reveals any mathematical regularity for the value of π. Against the background of the philosophies of antiquity, one can appreciate the great disappointment which this fact caused to mathematicians and philosophers. This failure regarding the outstanding ratio of the most perfect curve to conform to any pattern of mathematical regularity was considered as a blemish upon the divine world order, and never accepted as the ultimate answer.

The anticipations of antiquity regarding π finally proved justified, but the solution was found only as recently as 360 years ago. The value of π was expressed for the first time in a regular mathematical pattern

in 1592 by the great French mathematician, François Viète (1540–1603), who found:

$$\pi = 2 \cdot \frac{1}{\sqrt{½} \cdot \sqrt{½ + ½\sqrt{½}} \cdot \sqrt{½ + ½\sqrt{½ + ½\sqrt{½}}} \cdots}$$

The denominator is an infinite product of expressions of square roots with a regular structure. The possibility of one such development suggests the possibility of other simpler ones; and actually, in 1655, John Wallace (1616–1703), an English mathematician, found:

$$\pi = 4 \cdot \frac{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10 \cdot 10 \cdot 12 \cdot 12 \cdots}{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot 11 \cdot 11 \cdot 13 \cdots}.$$

Here π is expressed by infinite products of numbers, this time in both numerator and denominator of a fraction, but without any roots. In the numerator we find the even numbers, in the denominator the odd numbers. Both appear in pairs with the exception of the first factor in the numerator. Only three years later, in 1658, Viscount Brounecker (1620–1684) expressed the value of π as a continued fraction:

$$\pi = 4 \cdot \cfrac{1}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cfrac{9^2}{2 + \cdots}}}}}}$$

which again shows complete regularity, the only varying figures being the squares of the odd numbers.

Progress was on the march. The same century brought the final presentation of π as the limit of an infinite series of the simple fractions made up of the odd numbers as their denominators and with alternating signs, the Leibnitz Series. The regularity which was impossible in decimal expressions of the value of π now became possible through an infinite series of common fractions. Actually, this expression in fractions was more in keeping with the work of the thinkers of antiquity than was that in decimals, which have been in use only since the sixteenth century. That the series is infinite (the transcendence of π was proved by F. Lindemann in 1882) makes the result even more dynamic.

The Leibnitz Series is a fruit of the calculus obtained by one of its

inventors. It is derived from expanding the function of arctangent according to Maclaurin's series.

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$
$$+ \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots.$$

The form for arctan x thus reads:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \cdots,$$

which converges for all values of x within the limits

$$-1 \leq x \leq 1.$$

Substituting $x = 1$ for an angle of 45° (in radians $45° = \pi/4$; $\tan 45° = 1$) we obtain:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \cdots$$

or

$$\pi = 4 \cdot \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \cdots\right).$$

The Leibnitz Series has not been surpassed in all subsequent history in point of its outstanding simplicity. The only later additions were devices for calculating larger numbers of decimals with less effort in the process of computation,— in other words, by finding means of developing π through faster convergencies.

By expanding the arcsine in the same way we obtain the formula:

$$\arcsin x = x + \frac{1}{2} \cdot \frac{1}{3}x^3 + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{5}x^5 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{7}x^7$$
$$+ \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{1}{9}x^9 + \cdots$$

which converges for all values of x within the limits of $-1 \leq x \leq 1$. Substituting $x = 1$, we obtain for arcsin 1, corresponding to an angle of 90°, or, in radians, to $\pi/2$, the formula:

$$\frac{\pi}{2} = 1 + \frac{1}{2} \cdot \frac{1}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{7} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{1}{9} + \cdots$$

a series which, though more complicated than the Leibnitz Series, converges faster. Further series show a still greater convergence—for

instance, that which Abraham Sharp used in 1717 to calculate the value of π to 72 decimal places:

$$\pi = 6 \cdot \frac{1}{\sqrt{3}} \cdot \left(1 - \frac{1}{3 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \frac{1}{3^4 \cdot 9} - \frac{1}{3^5 \cdot 11} + \cdots\right).$$

To find the value of π geometrically, Deinostratus (350 B.C.) used a curve called the Quadratrix. Its construction is shown in Figure 5. Above and below a horizontal base AB, a quarter of a circle with A as its center and AB as its radius is drawn and divided into equal parts. In Figure 5 there are 8 equal parts above and 8 below the base. Then the perpendicular radii are divided into the same numbers of equal parts as the quarters of the circles and through every point of division a horizontal line is drawn. After also adding a radius through each point of division on the circle, we start with the highest point C and mark the point where the next horizontal line and the next radius intersect, and then continue marking the intersection points of the second horizontal line and the second radius and so forth. The curve passing through these points is

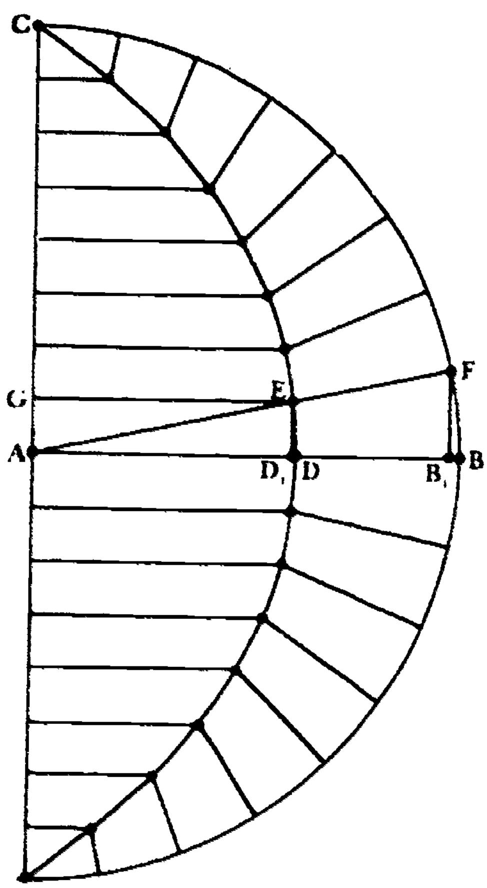


FIG. 5. Geometric construction of the value π.

the Quadratrix. Where it cuts the base AB is the point D and the ratio of the line segments AB and AD is[5]

$$\frac{AB}{AD} = \frac{\pi}{2}.$$

The geometric aspects of π lead to the famous problem of the quadrature of the circle, the task of constructing a square (quadratum) whose area equals the area of a given circle. The curve in Figure 5 also derives its name from this problem. An outstanding contribution to the quadrature of the circle was made by Archimedes who found that the area of a circle equals the area of a right triangle, one of whose legs equals the radius and the other the circumference of the circle. This discovery established an equality between the curved area of a circle and the area of a form bounded only by straight lines, and made possible the construction of the quadrature of a circle immediately upon straightening out its circumference. The latter task, so easily performed in actuality every time a wheel rolls over a road imprinting on the road its exact circumference with each revolution, has nonetheless been an age-long challenge to masters of geometric construction. Its complete solution is possible only by the use of higher curves. Numerous approximations of this geometric construction have been found, however, which for practical purposes represent a solution. Figure 6 shows the approximation constructed by Kochansky. Through the end-points of the vertical diameter are drawn two tangents to the circle. On each of these tangents a certain point is marked. On the lower tangent this point A is three times the length of the radius of the circle away from the point of tangency, while on the upper the point B is fixed at the intersection of the tangent with the prolonged radius drawn at an angle of 30° to the vertical diameter. The distance AB is then equal to π times the radius.[6] The

[5] The length of the arc BC being one-quarter of the circumference of a circle, is

$$\frac{2r\pi}{4} = r\frac{\pi}{2}.$$

Its ratio to the radius r is therefore $\pi/2$. The ratio of one-eighth of the arc BC to one-eighth of the radius is therefore also $\pi/2$. The length of the perpendicular from E to AB equals $ED_1 = AG$ which is by construction one-eighth of the radius AC; BF is 1/8 of BC. Therefore, the ratio BF to ED_1 is still $\pi/2$. What holds good for the eighths holds good for any other fraction. The smaller each part of the arc BC becomes, the closer it approaches the length of the perpendicular FB_1. Through the similarity of the triangles ΔAB_1F and ΔAD_1E we obtain the proportion:

$$\frac{AB_1}{AD_1} = \frac{FB_1}{ED_1}.$$

With an increasing number of points of division and the angle FAB decreasing in size, B_1 approaches B, D_1 approaches D, and the ratio FB_1/ED_1 the ratio FB/ED_1 which equals $\pi/2$.

[6] AB computed as the hypotenuse of the right triangle ABC, with its vertical leg $2r$, and its horizontal leg $3r$ minus the distance a, which is one-half the length of the base of an equilateral triangle with the altitude r ($a = r/\sqrt{3}$) is:

$$AB = \sqrt{(2r)^2 + \left(3r - \frac{r}{\sqrt{3}}\right)^2} = r\sqrt{4 + \frac{(3\sqrt{3}-1)^2}{3}} = r \cdot 3.14153.$$

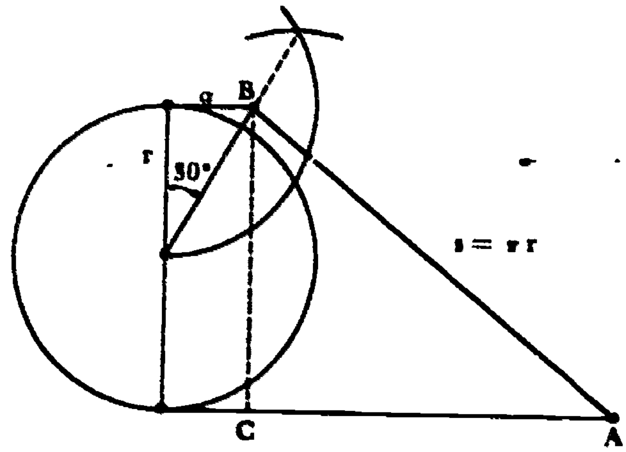


FIGURE 6. Construction by Kochansky.

approximation provides a difference of less than 0.0001, which lies beyond the graphical limit of precision of Figure 6.

With the help of Kochansky's construction, it is possible to effect the quadrature of the circle, as shown in Figure 7, in two steps. In the diagram to the left we recognize Kochansky's construction. The resulting distance is used as the base of a rectangle with an altitude equal to the radius of the circle. According to Archimedes, the area of the circle equals the area of a triangle whose base is the circumference of the circle and whose altitude is the radius. Therefore, it also equals the rectangle whose base is half the circumference of the circle and whose altitude is the radius. The next step consists in transforming the area of the rectangle into a square—a step accomplished as indicated in Figure 8. The rectangle *ADEF* is the same as the one in Figure 7. By construction,

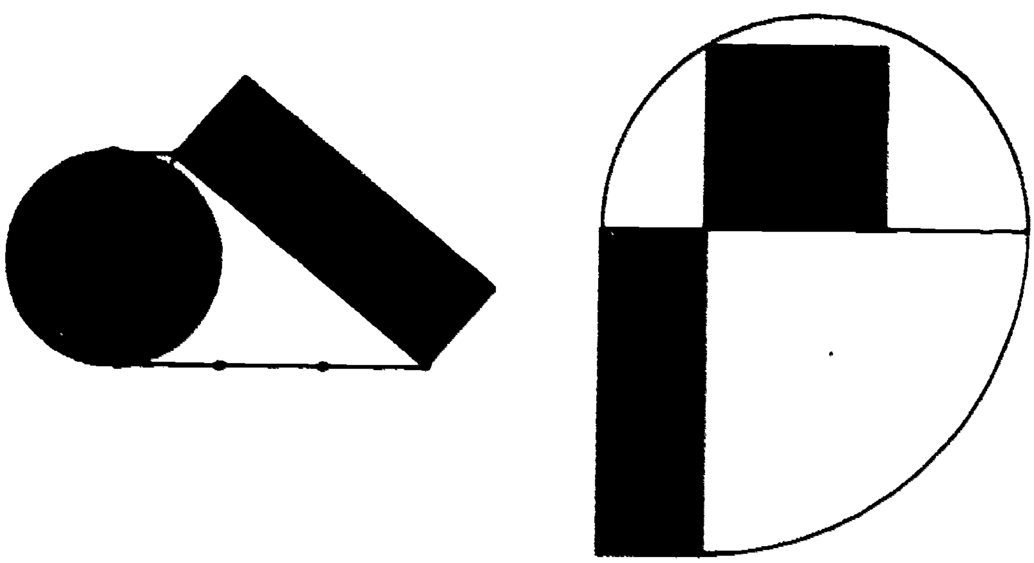


FIGURE 7. Construction of the quadrature of a circle.

DB is equal to *DE* and the intersection of the semicircle above *AB* with the prolongation of *DE* determines the point *C*. ΔABC is a right triangle with the altitude h. The area of the square with h as its side equals the area of the rectangle *ADEF*.[7]

The four areas which are marked in black in Figure 7 are equal to one another and show in their sequence from left to right the completion of the quadrature of the circle.

Finally, comparing the three great constants of mathematics, G, e, π:

$$G = 0.6180339887\cdots$$
$$e = 2.7182818284\cdots$$
$$\pi = 3.1415926535\cdots$$

in the form of continued fractions:

$$G = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}}$$

$$e = 1+1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cdots}}}}}$$

$$\pi = 4\,\cfrac{1}{1+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\cdots}}}}}.$$

[7] $AD = \dfrac{h}{\tan \measuredangle CAD}$;

$DE = DB = h \cdot \tan \measuredangle BCD$; $\measuredangle BCD = \measuredangle CAD$ (angles whose sides are perpendicular). Therefore

$$AD \cdot DE = \frac{h}{\tan \measuredangle CAD} \cdot h \tan \measuredangle CAD = h^2.$$

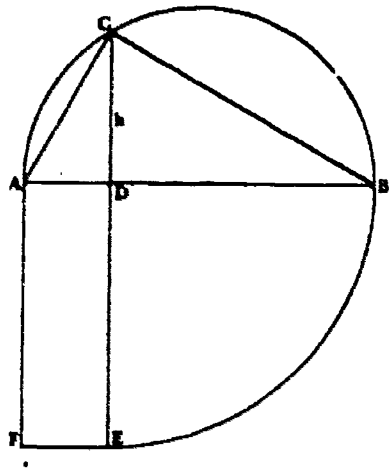


FIGURE 8.

G expresses itself through repetitions of the number 1 only, and e through repetitions of 1 and the powers of 2. In the fraction of π, the variable element is the series of the squares of the odd numbers.

There is an approximation between π and G which played a major role in the history of the investigations of the proportions of the Great Pyramid in Egypt:

$$\left(\frac{\pi}{4}\right)^2 = 0.6168 \cdots$$
$$G = 0.6180 \cdots.$$

Though the difference between the two values is only 0.0012, their closeness is merely incidental and has no basis in mathematical law. Neither is there any mathematical connection between G and e.

Different is the case with the constants π and e. Between them there is a distinct mathematical relationship. The expansion of e^x, according to MacLaurin's Series is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots$$

and that of $\sin x$ and that of $\cos x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots + (-1)^{n-1}\frac{x^{2n-2}}{(2n-2)!} + \cdots.$$

The two series for $\sin x$ and $\cos x$ together furnish all the terms of the series of e^x, with the only discrepancy in their signs, a difficulty which

does not exist for the hyperbolic sines and cosines. Their expansions have only positive terms:

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots + \frac{x^{2n-1}}{(2n-1)!}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots + \frac{x^{2n-2}}{(2n-2)!}.$$

Therefore it readily appears that $e^x = \sinh x + \cosh x$. An analogous result for the trignometric functions can be obtained if we substitute for x its product with the imaginary unit:

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\,\frac{x^3}{3!} + \frac{x^4}{4!} + i\,\frac{x^5}{5!} - \frac{x^6}{6!} - i\,\frac{x^7}{7!} + \cdots$$

and separate the real and imaginary terms. Thus, the result is:

$$e^{ix} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

or

$$e^{ix} = \cos x + i \sin x,$$

a formula which finds wide application in the solving of differential equations, particularly of those connected with all types of vibrations. Substituting $x = \pi$, we obtain:

$$e^{i\pi} = \cos \pi + i \sin \pi = -1 + i\,0 = -1$$

which results in the formula,

$$e^{\pi i} = -1.$$

It is this formula which prompted David Eugene Smith to use it in the mathematical credo placed in his library:

THE SCIENCE VENERABLE:

Voltaire once remarked — "One merit of poetry few will deny; it says more and in fewer words than prose." With equal significance we may say, "One merit of mathematics few will deny; it says more and in fewer words than any other science." The formula, $e^{\pi i} = -1$ expresses a world of thought, of truth, of poetry and of religious spirit, for "God eternally geometrizes."

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FOREWORD

Strangely enough, the C/D ratio is only one of the many properties of the number π, and is by no means the most meaningful property. It is known, for example, that if two numbers are written at random, the probability that they will be prime to each other is $6/\pi^2$. Or, consider another probability ratio: on a plane, a number of equidistant parallel lines are ruled, say at a distance d from one another, and then a stick of length k, where $k < d$, is dropped on a plane at random; the probability that the stick will fall in such a way as to lie across one of these lines is given by $2k/\pi d$. This can be proved theoretically as well as "corroborated" experimentally by recording the results of a very large number of trials.

About 1750, the celebrated Swiss mathematician Euler developed the analytic properties of the sine and cosine and established the relation

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

From this it is easy to see that, for $\theta = \pi$,

$$e^{i\pi} = \cos\pi + i\sin\pi.$$

But since

$$\cos\pi = -1 \text{ and } \sin\pi = 0,$$

we have

$$e^{i\pi} = -1.$$

This is but one of many ways in which π enters various branches of mathematical analysis.

Pi and Probability

WALTER H. CARNAHAN

One of the ancient weaknesses of men seems to be to take a chance and place a bet on its outcome. And one of his oldest scientific interests is that of the relation of diameter and circumference of a circle. In this brief article we shall call attention to the relation of this scientific interest to that of the observation of certain results of chance (but not the placing of bets).

Some two hundred years ago Buffon did an interesting experiment connecting π and probability. He tossed a needle onto a ruled surface, counted the tosses, and counted the number of times that the needle touched a line. Out of this experiment he found the value of π. This is a simple and interesting experience for high school pupils to repeat.

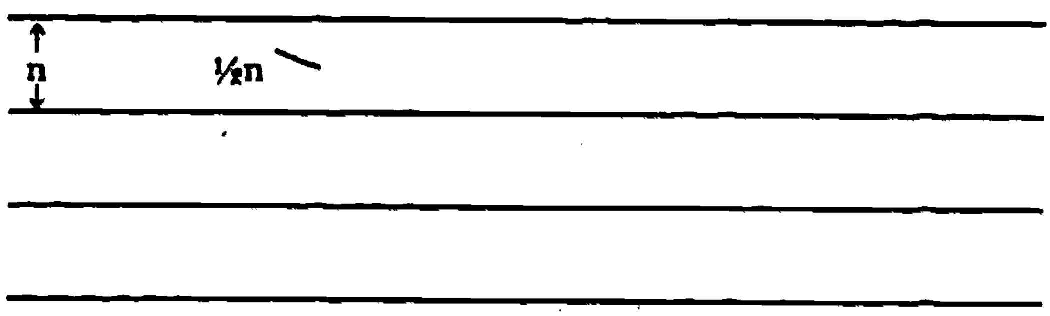

FIGURE 1

Rule off a board or paper with equally spaced parallel lines n units apart. Cut a wire whose length is $n/2$ units. (This length is not necessarily $n/2$ but is suggested as a convenient one.) Now toss the wire at random onto the ruled surface, count tosses T and contacts C. After fifteen minutes or more, divide T by C. The result is approximately equal to π.

The proof of this conclusion is not beyond the comprehension of a high school pupil. Suppose that the wire is bent into a circle; its circumference is $n/2$, and its radius is $n/4\pi$. Considered as a geometric line, the number of points on the needle is proportional to its length.* (The

* By the author's frank parenthetic admission, this is a very loose statement. The idea might be restated as follows: "the number of contacts is proportional to the length of the needle." Even this observation, based purely on intuition, leaves much to be desired by way of mathematical rigor. —EDITOR

philosophy underlying this statement might be debatable, of course.) Whether the needle is straight or bent, one point on it is just as likely to touch a line as is any other point. The shape of the needle will not affect the probability of any given point coming to rest on a line. Hence we can develop the discussion by assuming that the needle is bent into a circle. Since always two points on the circle will rest on the line if the line is in tangent or secant position, the probability of a one-point contact equals two times the probability of a secant relation.

A line will have a secant relation to the circle if the center of the circle is within radius distance of the line. The distance between two lines is n, and in the area between any two lines there are two areas $n/4\pi$ units wide in which the center of the circle could lie for a secant relation. Hence the probability of a secant relationship is $n:n/2\pi = 2\pi$. Therefore the probability of a one-point contact is π.

Now, the probability of contact when the needle is tossed is T/C. Hence $\pi = T/C$.

Another simple experiment for finding the value of π by using probability is tossing a coin onto a cross-ruled board. The distance between

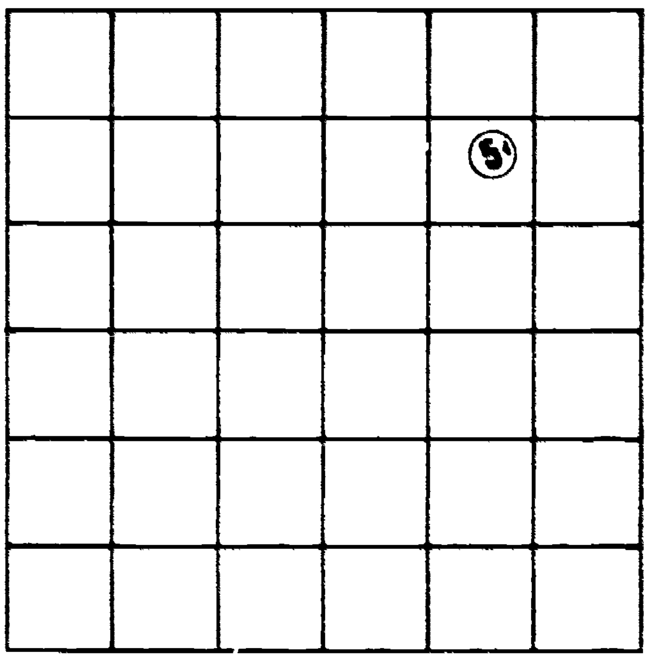

FIGURE 2

consecutive vertices of any square should be not less than the diameter of the coin; it may be greater than this. Toss the coin for fifteen minutes, count tosses T and count the number of times C the coin touches a vertex of a square. Multiply the area of a square by C, and divide this product by r^2T, r being the radius of the coin. This is approximately

equal to π. It is convenient to take s the side of the square equal to $4r$. If this is done, then $\pi = 16C/T$.

The proof of this is simple. The area of the part of any square in which the center of the coin can fall for contact is the area of the coin (four quadrants). The total area in which the center of the coin can lie is the area of the square itself. Hence the formula as derived from consideration of the law of probability.

In rationalizing the result of the coin-tossing experiment our attention was on the center of the circle. The coin itself merely served to determine the size of undrawn circles on the boar th contains the drawn squares. An alternative is to draw the circles a..d not draw the squares. The centers of the circles as shown in the figure are at the vertices of squares. The size of the circles or of the squares is not important so long as the circles do not overlap each other. For convenience let the radius r of each circle be 1 inch, and let the side s of each square be 4 inches. Toss darts at the board without aiming at any particular point on it. Count tosses T and the number of times C that a dart enters a circle. Divide $16C$ by T. This is approximately equal to π.

The reason for this is readily seen. Since the entire board is covered by squares (not drawn), the dart always enters a square. In every square there is a circle (in four quadrants) into which some of the darts will

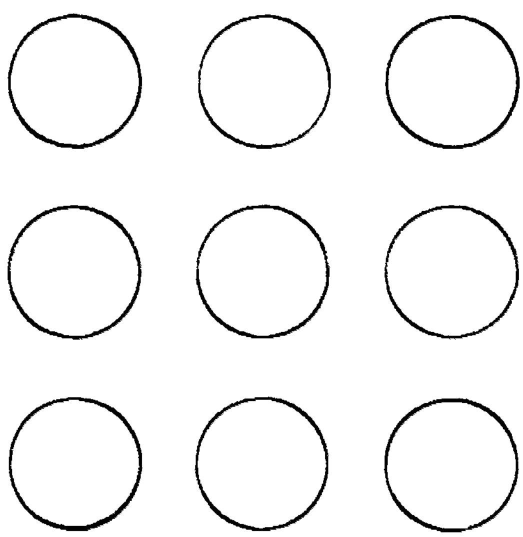

FIGURE 3

enter. The mathematical probability of a dart entering a circle is the ratio of the area of a circle to the area of a square, that is, $\pi r^2/s^2$. The experimental probability is C/T. Hence $\pi r^2/s^2 = C/T$, or $\pi = 16C/T$.

Any suitable device can be used to select the point on the board in the above experiment. If the board is level, a rolled marble would do, although with this device there is a difficulty of telling where the point of contact of marble and board is located. A tossed disc with a hole through the center is very convenient. One can use a rifle or air pistol if one aims at the board in general and not at any particular point on it. Small circles on a board placed at a great distance will help. (See Fig. 3.)

The preceding experiment can be varied by drawing a set of ellipses on rectangles rather than circles on squares. We shall not go through

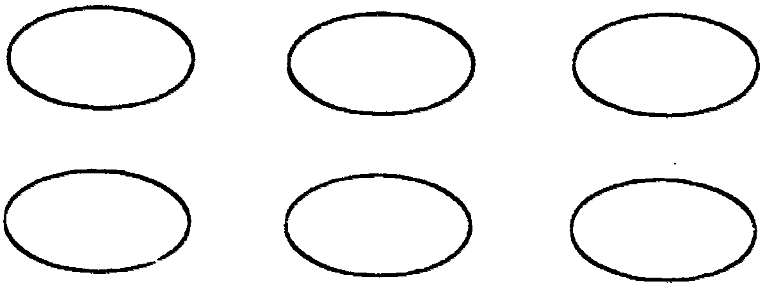

FIGURE 4

the details. We might repeat the familiar textbook statement that "This is left as an exercise for the student." If a and b are the semiaxes of the ellipses, and if $3a$ and $3b$ are sides of each rectangle, then $\pi = 9C/T$. (The area of an ellipse is πab.)

There are numerous possible variations of the devices suggested: One square in one circle; one circle in one square; one ellipse on one rectangle. One can even cut up the squares or circles and scatter the pieces so long as they do not overlap. Or one can cut the figures and arrange the pieces in patterns. The ratio of the areas is the essential consideration.

FOREWORD

Much of the mysticism and controversy surrounding the number π may be attributed to the many ill-fated attempts "to square the circle," that is, to construct a square that is exactly equal in area to the area of a given circle, using only the compass and a straightedge. If it were possible thus to square the circle, it would also be possible, with compass and straightedge only, to construct a line segment exactly equal in length to the value of π. The problem of constructing a straight line segment of length equal to that of a circumference is referred to as the "rectification of the circle."

The story of the many attempts to square the circle falls into three periods. The first period, from earliest Egyptian times to the middle of the seventeenth century, was characterized by the use of geometric methods. Men sought an exact construction of π by calculating the sides or areas of regular polygons inscribed and circumscribed in a circle.

The second period, from about 1650 to 1750, showed the influence of the newly invented calculus, and was characterized by the use of analytical methods. Men sought to express π analytically in terms of continued fractions, convergent series and infinite products. During this period great interest was aroused in laboriously computing the value of π to more and more decimal places. None of this activity, however, led to any further insight into the nature of the number π; it was not even known whether π was rational or irrational. However, one discovery was of significance: the relation between π and e, namely,

$$e^{i\pi} = -1,$$

developed by Euler about 1748.

The third period was devoted to an intense and profound study of the real nature of the number π. The cumulative efforts of the brilliant analysts J. H. Lambert (c. 1768), Legendre (c. 1794), Fourier (c. 1815), Liouville (c. 1840), and Hermite (c. 1873), culminated in Lindemann's proof (1882) that π is a transcendental number, that is, it cannot be the root of an algebraic equation.

It can be shown that a point p can be determined by means of compass and straightedge alone only if each of its coordinates is a root of an equation of some degree, a power of 2, of which the coefficients are rational functions of the coordinates of the points of the given data. This criterion leads to the conclusion that the problem of the rectification of the circle as defined above is possible only if π is a root of an algebraic

equation with rational coefficients, of that special class whose roots can be expressed by rational numbers or by numbers obtainable by successive extractions of square roots.

When it was finally proved that π is a transcendental number, and therefore not the root of an equation of the type stated in the above criterion, then it was shown once and for all that "squaring the circle" is impossible. Yet the tribe of would-be circle-squarers, totally unabashed and unconvinced, carries on, even to the present day.

Evolution of Pi: An Essay in Mathematical Progress from the Great Pyramid to Eniac

Panos D. Bardis

If a man were robbed by the river of a part of his land, he would come to Sesostris and tell him what had happened; then, the king would send men to inspect and measure the degree to which the land had been diminished, so that in the future it should pay in proportion to the tax imposed originally. In this way, it seems to me, geometry was born and came to Greece.

Herodotus, *Historiai*, II, 109.

Introduction

The Nile's contribution to mathematics, the Queen of the Sciences, has been followed by innumerable developments, many of which were more spectacular than the one described by Herodotus. And while our mathematical knowledge advanced, numerous other fields, including the social sciences, achieved a higher degree of progree as the Queen of the Sciences became their most invaluable handmaid. Of course, since the natural, social, and psychological worlds are complex, mysterious, and unfathomable, what we know at the present time still constitutes only an infinitesimal fragment of the boundless realm of truth. That is why the real scientist—not the charlatan—often makes statements similar to the Socratic "I only know one thing—that I know nothing" or Galileo's proverbial "I do not know," and even reminds us of the genuine and spontaneous humility with which little Jo, in Dicken's *Bleak House*, constantly exclaims, "I don't know nothink about nothink at all." And his humility is reinforced considerably by the realization that the progress of his science, like that of every other field, has often been similar to the movement of a glacier, due to myriads of incorrect theories, unsound assumptions, and inadequate methods. Indeed, even in the history of mathematics we find many theories which have proved to be about as erroneous as Homer's famous reply to Hesoid concerning the number of Achaeans who went to Ilium, namely, "There were fifty hearths, and at each hearth were fifty spits, and on each spit were fifty joints of meat; and there were three times three hundred Achaeans around each joint" (Alcidamas, *Peri Homerou kai Hesiodou*, 319).

This last point may be partly illustrated by presenting some of the most important stages in the evolution of what William Jones has

called π. Before doing so, however, it would be interesting to mention one of the most amusing mathematical paradoxes dealing with this value. According to Augustus De Morgan's *Budget of Paradoxes* (1872), Comte de Buffon (1707–88) asserted that, as Laplace proved later, the value of π may be calculated roughly by means of this strange experiment: after drawing two parallel lines, at distance D apart, on a floor, throw a needle of length $L < D$. The probability that the needle will cross one of the parallel lines will then be $2L/D_\pi$. Indeed, in 1855 Ambrose Smith of Aberdeen performed this experiment with 3,204 trials and found that $\pi = 3.1412$. "A pupil of mine," De Morgan also informs us, "made 600 trials with a rod of the length between the seams, and got $\pi = 3.137$."

At any rate, while reading numerous ancient, medieval, and modern mathematical and nonmathematical classics, I found many interesting passages dealing with the circle ratio. Of these, I have summarized the most important ones for the present article, in which the items included have been classified both geographically and chronologically, as follows:

Japan. Yoshida Shichibei Koyu or Mitsuyoshi (1598–1672), who wrote *Jinko-ki* (Small Number, Large Number, Treatise), the first great Japanese work dealing with arithmetic, gave π as 3.16. Imamura Chisho, on the other hand, Mori's famous pupil, in his *Jugai-roku* (1639) states that $\pi = 3.162$. The same value was given by Yamada (1656), Shibamura (1657), and Isomura (1660), while Muramatsu (1663), in the fourth book of his *Sanso*, which deals with the mensuration of the circle, gives π as 3.14, unaware of the fact that he had actually calculated the first eight figures of this value. Later, Nozawa (1664) and Sato (1666) also asserted that $\pi = 3.14$.

In the seventeenth century, Japan's greatest mathematician was Seki, who discovered a type of calculus known as *yenri* (circle principle), the main problem of which may be represented by Oyama Shokei's (1728) formula,

$$a^2 = 4dh\left[1 + \sum_{1}^{\infty} \frac{2^{2n+1}(n!)^2}{(2n+2)!} \cdot \left(\frac{h}{d}\right)^n\right],$$

where a stands for the length of the arc, d for the diameter, and h for the height of the arc. On the basis of this principle, in the next century, Matsunaga Ryohitsu calculated π to 50 figures. At about the same time, Takuma Genzayemon of Osaka employed the perimeters of polygons of 17,592,186,044,416 sides and computed π to 25 decimal places. Then, in 1769 Arima Raido, Lord of Kurume, published his *Shuki Sampo* in which he gives π to 29 figures by stating that

$$\pi = \frac{42822\ 45933\ 49304}{13630\ 81215\ 70117}.$$

In the sixth book of another mathematical treatise, Aida's *Sampo Kokon Tsuran* (1795), we find that

$$\frac{\pi}{2} = 1 + \frac{1!}{3} + \frac{2!}{3\cdot 5} + \frac{3!}{3\cdot 5\cdot 7} + \frac{4!}{3\cdot 5\cdot 7\cdot 9} + \cdots.$$

China. Ch'ang Hong (78–139), Emperor An-ti's minister and astrologer, believed that $\pi = \sqrt{10}$, while Wag Fan (229–67), an astronomer, asserted that

$$\pi = \frac{142}{45}.$$

Toward the end of the third century, what is known as "Chih's value of π" was given by Liu Chih as $3\frac{1}{8}$. One and a half centuries later, Wu, a geometer, stated that $\pi = 3.1432+$. Tsu-Chung-Chih (430–501), however, gave an "inaccurate value," 22/7, and an "accurate value," 355/113, also stating that π is found between 3.1415926 and 3.1415927. Another geometer, Men (c. 575), took π as 3.14, while in the thirteenth century, Ch'in Kiu-shao, in his *Su-shu* or *Nine Sections of Mathematics*, gave π as 3, 22/7, and $\sqrt{10}$. Finally, Ch'en Chin-mo (c. 1650) took π as 3.15025.

India. After 327 B.C., when Alexander the Great invaded this country, India's mathematicians were influenced considerably by the mathematical works of the Greeks. This influence is partly indicated by many scientific terms which the Indians borrowed from various Hellenic writings. The Hindu scientists, for instance, used the word *kendra* for center—from the Greek *kentron*—and *jamitra* for diameter—from the Greek *diametros*. Long before Alexander's time, however, India had many brilliant mathematicians. Among them was Baudhayama (c. 500 B.C.), who, in one of the *Sulvasutras* (Rules of the Cord), suggests that the construction of a circle equivalent to a square may be achieved by increasing half the length of one side by one-third of the difference between itself and half the length of the diagonal, which means that $\pi = 3.088$.

A thousand years later, in the celebrated *Paulisa Siddhanta*, a treatise on trigonometry, we find that $\pi = \sqrt{10}$. At about the same time, Aryabhata the elder of Kusumapura, the City of Flowers, wrote his famous *Aryastasata* in which the volume of the sphere is inaccurately given as $\pi r^2\sqrt{\pi r^2}$, which leads to $\pi = 16/9$, undoubtedly a distortion of the Egyptian value

$$\left(\frac{16}{9}\right)^2,$$

computed by Ahmes. Then, in the sixth century, Aryabhata the younger wrote the *Ganita*, a poem in 33 couplets, the fourth of which deals with π as follows: "Add 4 to 100, multiply by 8, and add again 62,000; the

result is the approximate value of the circumference of a circle of which the diameter is 20,000." Accordingly,

$$\pi = \frac{62{,}832}{20{,}000}$$

or 3.1416. In the next century, Brahmagupta (c. 628) used two values for π, the "practical" one or 3, and the "neat" one or $\sqrt{10}$. This last value was also employed by Mahavira the Learned (c. 850) in his *Ganita-Sara* (Compendium of Calculation), as well as by Sridhara (eleventh century) in his *Trisatika* (300 Couplets).

Babylonia. As early as 2100 B.C., the mathematicians of Babylonia dealt with the circle ratio, but took it as 3.

Hebrews. The value of 3 is also found in the Talmud, a collection of Hebrew books dealing with ceremonial regulations and laws, as well as in two passages of the Old Testament. According to the first of these, "And he made a molten sea, ten cubits from the one brim to the other: it was round all about, and his height was five cubits: and a line of thirty cubits did compass it round about" (I Kings, VII, 23). Similarly, the second passage reads as follows: "Also he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a lime of thirty cubits did compass it round about" (II Chronicles, IV, 2).

Egypt. One of the many fascinating theories dealing with the mystery of the Great Pyramid, which was built 5,000 years ago, is analyzed by Abbé Moreux in his *La science mystérieuse des Pharaons* (Paris, 1923, pp. 28–29), where he states: "Additionnons en effet les quatre côtés de la base du monument dont la valeur était primitivement de $232^{m}805$; nous aurons pour le périmètre $931^{m}22$: Soit: $4 \times 232^{m}805 = 931^{m}22$. Divisons maintenant la longueur de ce périmètre par 2 fois la hauteur de la pyramide qui était *à* l' époque de sa construction de $148^{m}208$, nous trouverons la valeur de π." This means that

$$\pi = \frac{931^{m}22}{2 \times 148^{m}208} = 3.1416.$$

About one thousand years later, the famous Golenischev or Moscow Papyrus, which includes 25 mathematical problems, gave π as

$$\left(\frac{16}{9}\right)^2.$$

The same value is found in Ahmes's (moon-born; 1650 B.C.) *Directions for Obtaining the Knowledge of All Dark Things*, a book included in the Rhind Collection of the British Museum and containing 80 problems. Indeed, according to the fiftieth problem, the area of a circle may

be calculated by deducting from the diameter 1/9 of its length and squaring the remainder, which means that

$$\pi = \left(\frac{16}{9}\right)^2 = 3.1604.$$

Greece. Archimedes (287–212 B.C.) dealt with the circle ratio in one of his most famous works, *The Measurement of the Circle,* which consists of three mathematical propositions. The third of these propositions reads as follows: "The ratio of the circumference of any circle to its diameter is less than 3 1/7 but greater than 3 10/71." A close approximation was also given by Claudius Ptolemy in A.D. 150 in his great treatise, *Megale syntaxis tes astronomias* (VI, 7), where he states that

$$\pi = 3 \cdot 8' \cdot 30'' = 3 + \frac{8}{60} + \frac{30}{3600} = 3.141{,}666 \cdots.$$

Michael Constantine Psellus (1020–1110), however, the Neoplatonist whom the Byzantine emperors called *Philosophon hypatos* (Prince of Philosophers), took π as $\sqrt{8}$.

Italy. Pietro della Francesca, an Italian painter, in 1475 published his *De corporibus regularibus,* in which he discussed his famous problem of the regular octagon by stating that "Diameter circuli qui circumscribit octagonum est 7," and using π as 22/7.

Switzerland. Leonhard Euler, one of the greatest mathematicians, popularized the symbol π in 1737, but this was not the first time that the circle ratio was represented by the sixteenth letter of the Greek alphabet.

France. François Vieta (1540–1603), a great expert in deciphering the cryptic writing of diplomatic documents and one of the first to introduce letter symbols in algebra, employed one of the earliest methods of computing the value of π by means of infinite products. He thus stated that

$$\frac{2}{\pi} = \sqrt{\tfrac{1}{2}} \cdot \sqrt{\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}}} \cdot \sqrt{\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}}}} \cdots.$$

More than a century later (1719), De Lagny gave π to 127 places.

Germany. Albertus de Saxonia (1325–90), the bishop of Halberstadt, considered π equal to 3 1/7. On the other hand, in the sixteenth century, Ludolf van Ceulen became famous by devoting many years to the calculation of the circle ratio, which he gave to 20 decimal places in his *Van den Circkel,* and later to 35 in his *De arithmetische en geometrische fondamenten.* This achievement was regarded so important that the value of π was cut on his tombstone in St. Peter's churchyard at Leyden, and, in addition, the circle ratio was named "Ludolf's number" — this term is often employed by mathematicians even at the present time. Then

π was computed to 140 places by Georg Vesa in 1793, to 205 by Zacharias Dase in 1844, and to 250 by T. Clausen in 1847, while in 1882 F. Lindemann proved the transcendence of π, a discovery that led to Kronecker's well-known question, "Of what value is your beautiful proof, since irrational numbers do not exist?"

Netherlands. Adriaen Anthoniszoon or Metius (1543–1620) took 355/113 as the value of π, and Adriaen van Roomen of Louvain (1561–1615), in his *Ideae mathematicae,* gave π to 17 decimal places.

England. Finally, John Wallis (1616–1703), a brilliant cryptologist and one of the founders of the Royal Society, gave one of the first values of the circle ratio involving infinite products. Thus, his well-known product for π is

$$\frac{\pi}{2} = \frac{2}{1}\,\frac{2}{3}\,\frac{4}{3}\,\frac{4}{5}\,\frac{6}{5}\,\frac{6}{7}\cdots\frac{2k}{2k-1}\,\frac{2k}{2k+1}\cdots.$$

Another founder of the Royal Society, Lord Brouncker (1620–84), through his work on the quadrature of the circle, discovered that

$$\frac{4}{\pi} = \frac{1}{1+}\,\frac{1^2}{2+}\,\frac{3^2}{2+}\,\frac{5^2}{2+}\cdots.$$

Then, in 1705 Abraham Sharp computed π to 72 places, while in 1706 an important step was taken by William Jones, who, on page 263 of his *Synopsis palmariorum matheseos,* for the first time expressed the circle ratio by means of the symbol π. The passage referring to this matter reads as follows: ". . . in the *Circle,* the Diameter is to the Circumference . . . 3.14159, &c. $= \pi$" (π is the initial letter of the Greek word *periphereia,* which means periphery or circumference)." In the same year, John Machin, a professor of astronomy in London, calculated π to 100 decimal places, and, finally, in 1853 W. Shanks gave 707 figures!

Before closing, I should also add this last item: a few years ago, thanks to various modern scientific developments, the calculating machine known as ENIAC took only 70 hours and correctly computed π to 2,035 decimal places!

For Further Reading and Study

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